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Critical Values in a Long-range Percolation on Spaces Like Fractals

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On a certain class of general discrete spaces including fractals, we consider a model in which each pair of distinct points is connected by a random bond. The main question we are concerned is whether a connected component consisting of infinitely many points exists or not. This depends on the choice of parameters in the connecting probabilities, and the aim of this paper is to find thresholds of the parameters.

KEY WORDS: Long-range percolation, phase transition, cluster, Fractals

1. INTRODUCTION AND RESULTS

The problem of a long-range percolation was introduced by Schulman in Ref. 17 and it is well-studied on a d-dimensional integer lattice $\mathbf{Z}^d$ by now, see below. In this paper, we consider the long-range percolation in some general settings. We shall discuss on the space X, instead of $\mathbf{Z}^d$.

Let X be a countable infinite set, and let ρ be a map from $X \times X$ to $\mathbf{R}_+ = [0, \infty)$, which satisfies the following two conditions;

- $\rho(x, y) = 0 \leftrightarrow x = y \ (x, y \in X)$,
- $\rho(x, y) = \rho(y, x)$.

For example, a metric on X satisfies these two conditions.

Each unoriented pair of distinct points $x, y \in X$ is connected by an unoriented bond with probability $p(x, y) = p(y, x) \in [0, 1)$, independently of other pairs. In other words, we consider the following probability space Ω and probability

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measure $P_{\mathbf{p}}$ on it; we denote $\mathbf{p} = \{p(x, y)\}_{x,y \in X, x \neq y}$.

$$\Omega = \prod_{(x,y)\in X\times X, x\neq y} \{0, 1\}, \qquad P_{\mathbf{p}} = \prod_{(x,y)\in X\times X, x\neq y} \mu_{xy},$$

where $\mu_{xy}(1) = p(x, y)$, $\mu_{xy}(0) = 1 - p(x, y)$, and the products are taken over all unoriented pairs (x, y). The state 1 means that x and y are connected by a bond $\langle x, y\rangle$ $(= \langle y, x\rangle)$, while the state 0 means that there is no bond connecting x and y.

This paper especially studies the case where $\mathbf{p}$ satisfies the condition

$$\lim_{\rho(x,y)\to\infty} \frac{p(x, y)}{\beta\rho(x, y)^{-\alpha}} = 1$$

for some $\alpha, \beta > 0$.

For given ρ, the main question we are concerned with is to classify the relation between $\mathbf{p}$ and the probability of an ∞-cluster, i.e., a connected component consisting of infinitely many points of X, to exist.

When $X = \mathbf{Z}$ and $\rho(x, y) = |x - y|$, it is known that the phase transitions occur at $\alpha = 1, 2$, and $\beta = 1$. More precisely,

(1) When $\alpha \leq 1$, under a certain aperiodicity condition, all points are connected with probability 1.
(2) When $1 < \alpha < 2$ or "$\alpha = 2$ and $\beta > 1$," one can choose $\mathbf{p}$, for which an ∞-cluster exists with probability 1, and one can also choose $\mathbf{p}$, for which the probability that an ∞-cluster exists is 0.
(3) When "$\alpha = 2$ and $\beta \leq 1$," or $2 < \alpha$, the probability that an ∞-cluster exists is always 0.

For details, see Refs. 1, 11, 16 and others. When $X = \mathbf{Z}^d$ $(d \geq 2)$ and $\rho(x, y) = |x - y|_{\mathbf{Z}^d}$, by the result of (nearest-neighbor) bond percolation on $\mathbf{Z}^d$, an ∞-cluster may exist for any α, β. Among recent studies on long-range percolation, random walks on ∞-clusters are discussed in Ref. 3, and chemical distances are studied by Refs. 2, 4–8, 10. The long-range percolation on random sets in $\mathbf{R}^d$ is considered in Ref. 9.

In this paper, we extend the problem on more general spaces, including fractals. The conditions we always assume on ρ is the following;

(A1) There exist $c_i > 0 (1 \leq i \leq 4), a, b > 1$, and for each $x \in X$, there exists $\{B(x, n)\}_{n=0}^{\infty}$, a sequence of subsets of X, such that

- $x \in B(x, 0) \subset B(x, 1) \subset B(x, 2) \subset \cdots$,
- $\cup_{n=0}^{\infty} B(x, n) = X$,
- $c_1 a^n \leq \sup_{y,z\in B(x,n)} \rho(y, z) \leq c_2 a^n$ $(n \geq 0)$,
- $c_3 b^n \leq |B(x, n)| \leq c_4 b^n$ $(n \geq 0)$,

where $|B(x,n)|$ stands for the number of points in $B(x,n)$. We denote

$$D = \frac{\log b}{\log a}.$$

We note that $\{B(x,n)\}_{n=1}^{\infty}$ and a, b may not be uniquely determined. In the following discussions, we consider with some fixed $\{B(x,n)\}_{n=1}^{\infty}$. Under some additional conditions, $\alpha = D, 2D$ may become critical values in certain sense in our setting, though the rigorous proof for the region $D < \alpha \leq 2D$ is still missing.

We prepare some further notations. For $x, y \in X$, $x \sim y$ means there exists a bond $\langle x, y\rangle$, and $x \leftrightarrow y$ means x and y are connected (i.e., for some positive integer n and for some $x_0, x_1, \ldots, x_n \in X, x = x_0 \sim x_1 \sim \cdots \sim x_n = y$). For $A, B \subset X$, $A \sim B$ means $x \sim y$ for some $x \in A$, $y \in B$. We define $A \leftrightarrow B$ similarly. We denote

$$C(x) = \{x\} \cup \{y \in X | y \leftrightarrow x\},$$

a connected component containing x, and

$$P_\infty = P_{\mathbf{p}}\left[\bigcup_{x \in X}\{|C(x)| = \infty\}\right],$$

the probability that an ∞-cluster exists. By Kolmogorov's $0-1$ law, P_∞ is either 0 or 1.

In the remaining of this section, we formulate main results of this paper and give examples covered by our results. In Sec. 2, we shall give proofs of Theorems.

We require the following condition (A2) in Theorem 1.1.

(A2) Let $\{B(x,n)\}$ be the same as in (A1). For any $x \in X$, one can find $\{x^{(n)}\}_{n=0}^{\infty}$, such that $x^{(n)} \in R(x,n) \equiv B(x,n) \backslash B(x,n-1)$, and for any n, there exists a bijective map from $B(x,n)\backslash\{x\}$ to $B(x,n)\backslash\{x^{(n)}\}$, which preserves ρ.

Theorem 1.1. *We assume $\alpha < D$, and let $\mathbf{p}$ satisfies that $p(x,y)$ is nonincreasing for $\rho(x,y)$ $(x, y \in X)$. Then, under (A1),(A2) for ρ, we have*

$$P_{\mathbf{p}}\left[\bigcap_{x \in X}\{C(x) = X\}\right] = 1.$$

Theorem 1.2. *We assume $\alpha = D$. Then, under (A1) for ρ, for any $x \in X$,*

$$P_{\mathbf{p}}[\#\{y \in X | y \sim x\} = \infty] = 1.$$

We require the following condition (A3) in Theorems 1.3 and 1.4.

(A3) Let $\{B(x, n)\}$ be the same as in (A1). There exists a positive constant c_5 (depending only on ρ) and for all $x \in X$ and all $n \geq 0$, the following holds.

$$c_5 a^n \leq \inf\{\rho(y, z) | y \in B(x, n), z \in R(x, n+1)\}.$$

Theorem 1.3. *We assume $\alpha > D$. Then, under (A1),(A3) for ρ, we can choose* $\mathbf{p}$*, for which $P_\infty = 0$.*

Theorem 1.4. *We assume $\alpha > 2D$. Then, under (A1),(A3) for ρ, $P_\infty = 0$.*

Before giving some examples covered by our results, we review the definitions of some fractals. See Refs. 15, 18, 19, for example, for more details for fractal lattices and bond percolation on them.

(1) (Sierpinski gasket lattice.) Let $\mathbf{0} = (0, 0), u_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), v_0 = (1, 0) \in \mathbf{R}^2$, and F_0 be a graph which consists of the vertices and edges of the triangle $\triangle \mathbf{0} u_0 v_0$. We define

$$F_{n+1} = F_n \cup (F_n + u_n) \cup (F_n + v_n),$$

where $u_n = 2^n u_0$, $v_n = 2^n v_0$. We call $F = \cup_{n=0}^\infty F_n$ the Sierpinski gasket lattice.

(2) (Sierpinski carpet lattice.) Let G_0 be a graph corresponding to a rectangle with vertices $(0, 0), (0, 1), (1, 0), (1, 1) \in \mathbf{R}^2$. For $T = \{(i, j) \in \mathbf{Z}^2 | 0 \leq i, j \leq 2, (i, j) \neq (1, 1)\}$, we define

$$G_{n+1} = \bigcup_{(i,j) \in T} \{G_n + (i3^n, j3^n)\}.$$

We call $G = \cup_{n=0}^\infty G_n$ the Sierpinski carpet lattice.

(3) (The space like Cantor set.) As an example for Theorems 1.3, 1.4, we also consider the space like Cantor set $H \subset \mathbf{Z}$, although it is not a connected graph. Let $H_0 = \{0, 1\}$, and

$$H_{n+1} = H_n \cup (H_n + 3^n \times 2).$$

We define $H = \cup_{n=0}^\infty H_n$.

Example 1.5. When X is the vertex set of the Sierpinski gasket lattice and ρ is a graph metric on the lattice, (A1) holds by regarding the vertex set of F_n as $B(\mathbf{0}, n)$. By the self-similarity, we can also find appropriate $\{B(x, n)\}_{n=0}^\infty$ for any point x. We note that, in our case, $B(x, n)$ does not stand for a ball with radius n, but the "n-th stage" set in constructing the lattice as above. In this case, (A1) is satisfied with $a = 2$, $b = 3$, and $D = \frac{\log 3}{\log 2}$ coincides with the Hausdorff dimension of the lattice. Furthermore, by noticing that $F_n \backslash \{(0, 0)\}$ is isomorphic to $F_n \backslash \{(2^n, 0)\}$ for any n, (A2) is also satisfied. Here, (A3) does not hold.

Example 1.6. When X is the vertex set of the Sierpinski carpet lattice and ρ is a graph metric on the lattice, (A1), (A2) hold by regarding the vertex set of G_n as $B(\mathbf{0}, n)$, in the same way as Example 1.5. In this case, $a = 3, b = 8$, and $D = \frac{\log 8}{\log 3}$. (A3) does not hold.

Example 1.7. When X is the space like Cantor set and ρ is a metric induced by the metric on $\mathbf{Z}$, (A1), (A2) holds by regarding H_n as $B(\mathbf{0}, n)$. In this case, $a = 3$, $b = 2$, and $D = \frac{\log 2}{\log 3}$. Now, (A3) is also satisfied by noticing that the distance between H_n and $H_{n+1} \backslash H_n$ becomes large at exponential order as $n \to \infty$.

Example 1.8. In the above construction of the Sierpinski carpet lattice, now, we redefine G_n by taking $T = \{(i, j) \in \mathbf{Z}^2 | 0 \le i, j \le 2\}$ instead of $T = \{(i, j) \in \mathbf{Z}^2 | 0 \le i, j \le 2, (i, j) \neq (1, 1)\}$. Then, G becomes the quarter space on $\mathbf{Z}^2$. When X is the quarter space on $\mathbf{Z}^2$ and $\rho(x, y) = |x - y|_{\mathbf{Z}^2}$, (A1), (A2) holds by regarding the vertex set of G_n as $B(\mathbf{0}, n)$. In this case, $a = 3$, $b = 9$, and $D = 2$. Here, (A3) is not satisfied. Theorems 1.1, 1.2 are also true for $X = \mathbf{Z}^2$, as the case of the quarter space. For general $\mathbf{Z}^d$ ($d \ge 1$), we can discuss in the same manner, and D coincides with d.

Remark 1.9. For Theorem 1.1, the corresponding results on $\mathbf{Z}^d$ or half spaces are studied in Refs. 12, 13, 14 etc. Theorem 1.1 covers the spaces without translation invariance, although the case $\alpha = D$ is excluded.

2. PROOF OF THEOREMS

Proof of Theorem 1.1. We prove Theorem 1.1 following the idea of Ref. 13. For $x, y \in X$, $x \neq y$, and $n \ge 0$, we denote

$$A_n = \{\omega \in \Omega | x \leftrightarrow y, \text{in} B(x, n)\}$$

(Let $A_n = \emptyset$ if $y \in B(x, n)^c$.) We shall show that $\lim_{n\to\infty} P_{\mathbf{p}}[A_n] = 1$. To this end, we estimate

$$\begin{aligned} P_{\mathbf{p}}\left[A_n^c\right] &= P_{\mathbf{p}}[x \not\leftrightarrow y, \text{in } B(x, n)] \\ &\le P_{\mathbf{p}}[\{\omega \in \Omega | \omega' \in \{x \not\leftrightarrow y, \text{in } B(x, n)\}\}] \\ &\le P_{\mathbf{p}}\left[\{\omega' \in \{x \not\leftrightarrow y, \text{in } B(x, n)\}\} \cap \bigcup_{i \in B(x,n)\backslash\{x,y\}} \{x \sim i\}\right] \\ &\quad + P_{\mathbf{p}}\left[\bigcap_{i \in B(x,n)\backslash\{x,y\}} \{x \not\sim i\}\right] \\ &\equiv I_1 + I_2, \end{aligned}$$

where ω' is a configuration which is made by removing the bond $\langle x, y\rangle$ (if it exists) from ω. First, we estimate the second term I_2.

Lemma 2.1. *When $\alpha \leq D$, the second term I_2 converges to* 0 *as $n \to \infty$.*

Proof: This is elementary, because I_2 is equal to and estimated as

$$I_2 = \prod_{i\in B(x,n)\setminus\{x,y\}} \{1 - p(x,i)\} \leq \prod_{i\in B(x,n)\setminus\{x,y\}} \exp\{-p(x,i)\}$$
$$= \exp\left\{-\sum_{i\in B(x,n)\setminus\{x,y\}} p(x,i)\right\},$$

and we note

$$\lim_{n\to\infty} \sum_{i\in B(x,n)\setminus\{x,y\}} p(x,i) = \sum_{i\in X\setminus\{x,y\}} p(x,i)$$
$$= \sum_{n=0}^{\infty} \sum_{i\in R(x,n), i\neq x,y} p(x,i)$$
$$\geq \sum_{n=0}^{\infty} c\beta(ba^{-\alpha})^n = \infty,$$

when $b \geq a^\alpha$ and the result follows. □

We return to the proof of Theorem 1.1. We write the event

$$E = \{x \sim y\} \cap \{\omega' \in \{x \not\leftrightarrow y, \text{ in } B(x,n)\}\} \cap \bigcup_{i\in B(x,n)\setminus\{x,y\}} \{x \sim i\}.$$

Then, the first term I_1 equals to

$$p(x,y)^{-1} P_{\mathbf{p}}[E] \leq p(x,y)^{-1} P_{\mathbf{p}}[X_n{}' > X_n]$$
$$= p(x,y)^{-1} P_{\mathbf{p}}[X_n{}'' > X_n].$$

Here, X_n, X_n', X_n'' stand for the number of connected components in $B(x,n)$, in $B(x,n)\setminus\{x\}$, and in $B(x,n)\setminus\{x^{(n)}\}$, respectively. We have used (A2) in the last equality. Now, we prepare the following two lemmas. These can be proved easily, but do not contain the case $\alpha = D$.

Lemma 2.2. *When $\alpha < D$,*

$$P_{\mathbf{p}}\left[\bigcup_{n=0}^{\infty}\bigcap_{k=n}^{\infty}\{X_k \geq X_{k+1}{}''\}\right] = 1.$$

Lemma 2.3. *When $\alpha < D$,*

$$P_{\mathbf{p}}\left[\bigcup_{n=0}^{\infty}\bigcap_{k=n}^{\infty}\{X_k{}'' \geq X_k\}\right] = 1.$$

Proof of Lemma 2.2. We have

$$\begin{aligned}
P_{\mathbf{p}}[X_n < X''_{n+1}] &\leq P_{\mathbf{p}}\left[\bigcup_{i\in R(x,n+1)\setminus\{x^{(n+1)}\}} \{i \not\sim B(x,n)\}\right] \\
&\leq |B(x,n+1)| \sup_{i\in R(x,n+1)\setminus\{x^{(n+1)}\}} P_{\mathbf{p}}[i \not\sim B(x,n)] \\
&\leq c_2 b^{n+1} \sup_{i\in R(x,n+1)} \prod_{j\in B(x,n)} \{1 - p(i,j)\} \\
&\leq c_2 b^{n+1} \sup_i \exp\left\{-\sum_{j\in B(x,n)} p(i,j)\right\} \\
&\leq c_2 b^{n+1} \sup_i \exp\left\{-|B(x,n)| \inf_{j\in B(x,n)} p(i,j)\right\} \\
&\leq c_2 b^{n+1} \exp\left\{-c(ba^{-\alpha})^n\right\}.
\end{aligned}$$

We have used the assumption for $\mathbf{p}$ in the last inequality. Therefore, $\sum_{n=1}^{\infty} P_{\mathbf{p}}[X_n < X''_{n+1}] < \infty$ when $ba^{-\alpha} > 1$, and this shows the conclusion by Borel–Cantelli lemma. □

Proof of Lemma 2.3. The proof is essentially the same as that of Lemma 2.2, and the calculation is indeed easier. We have

$$\begin{aligned}
P_{\mathbf{p}}[X_n{}'' < X_n] &= P_{\mathbf{p}}[x^{(n)} \not\sim \{B(x,n)\setminus\{x^{(n)}\}\}] \\
&\leq c \exp\{-(ba^{-\alpha})^n\},
\end{aligned}$$

and the result follows. □

We shall complete the proof of Theorem 1.1. From Lemmas 2.2 and 2.3, for some $\Omega_1 \subset \Omega$, $P_{\mathbf{p}}[\Omega_1] = 1$, we have $X_n \geq X_{n+1}{}''$ and $X_n{}'' \geq X_n$ for sufficiently large all n, in Ω_1.

On the other hand, if we assume $\limsup_{n\to\infty} P_{\mathbf{p}}[X''_n > X_n] = \varepsilon > 0$, it implies

$$\varepsilon = \limsup_{n\to\infty} E_{\mathbf{p}}[1_{\{X_n{}'' > X_n\}}] \leq E_{\mathbf{p}}\left[\limsup_{n\to\infty} 1_{\{X''_n > X_n\}}\right].$$

We have used Fatou's lemma in the last inequality. Then, for some $\Omega_2 \subset \Omega$, $P_{\mathbf{p}}[\Omega_2] > 0$, we have that $\limsup_{n\to\infty} 1_{\{X_n'' > X_n\}}$ is strictly positive (therefore, equals to 1) in Ω_2. So, in Ω_2, we have $X_n'' > X_n$ for infinitely many n.

Then, we can choose $\omega \in \Omega_1 \cap \Omega_2$, and $X_n(\omega)$ becomes non-positive for some n. On the other hand, X_n should be a positive integer, because it stands for the number of connected components. This is a contradiction. This means $\limsup_{n\to\infty} P_{\mathbf{p}}[X_n{}'' > X_n] = 0$, and the proof is complete. □

Proof of Theorem 1.2. For any $x \in X$,

$$\begin{aligned}\sum_{y\in X} P_{\mathbf{p}}[x \sim y] &= \sum_{n=0}^{\infty} \sum_{y\in R(x,n)} p(x,y) \\ &\geq \sum_{n=N}^{\infty} |R(x,n)| \inf\{p(x,y) | y \in R(x,n)\} \\ &\geq \sum_{n=N}^{\infty} c\beta(ba^{-\alpha})^n = \infty,\end{aligned}$$

when $b \geq a^{\alpha}$. We note the events $x \sim y$ are independent for each y, and the result for $\alpha = D$ follows by Borel–Cantelli lemma. □

Proof of Theorem 1.3. For any $x \in X$, it is enough to show that we can find $\mathbf{p}$ satisfying

$$E_{\mathbf{p}}[\#\{y \in X | y \sim x\}] \leq 1.$$

If the above estimate holds, we can see that $P_\infty = 0$ by comparing percolation cluster with Galton–Watson branching process with parameter 1. We consider the case of $p(x,y) = \beta\rho(x,y)^{-\alpha}$. Then, the left hand side is equal to and bounded by

$$\begin{aligned}\sum_{y\in X} P_{\mathbf{p}}[y \sim x] &= \sum_{n=0}^{\infty} \sum_{y\in R(x,n)} p(x,y) \\ &\leq \sum_{n=0}^{\infty} cb^n \sup\{p(x,y) | y \in R(x,n)\} \\ &\leq \sum_{n=0}^{\infty} cb^n \beta[\inf\{\rho(x,y) | y \in R(x,n)\}]^{-\alpha}\end{aligned}$$

$$\leq \sum_{n=0}^{\infty} c\beta (ba^{-\alpha})^n.$$

We have used (A3) in the last inequality. The summation in the right hand side converges if and only if $ba^{-\alpha} < 1$, and equals to $c\beta \frac{a^\alpha}{a^\alpha - b}$. We can therefore have a desired estimate by choosing $\beta \leq c^{-1}(1 - \frac{b}{a^\alpha})$. □

Proof of Theorem 1.4. We show that

$$\lim_{n\to\infty} P_{\mathbf{p}}[B(x,n) \not\sim B(x,n)^c] = 1$$

for any $x \in X$. For simplicity, we deal with the case of $p(x, y) = \beta\rho(x, y)^{-\alpha}$. We have

$$\begin{aligned} P_{\mathbf{p}}[B(x,n) \not\sim B(x,n)^c] &= \prod_{y\in B(x,n), z\in B(x,n)^c} \{1 - p(y,z)\} \\ &= \prod_{k=1}^{\infty} \prod_{y\in B(x,n), z\in R(x,n+k)} \{1 - p(y,z)\} \\ &\geq \prod_{k=1}^{\infty} [1 - \beta\{\inf_{y,z} \rho(y,z)\}^{-\alpha}]^{|B(x,n)||B(x,n+k)|}, \end{aligned}$$

where the infimum in the right hand side is taken over all $y \in B(x,n)$, $z \in R(x, n+k)$. By (A3), the part of $k = 1$ in the product converges to 1 as $n \to \infty$, when $\alpha > 2D$;

$$\begin{aligned} &\{1 - \beta\{\inf\{\rho(y,z) \mid y \in B(x,n), z \in R(x,n+1)\}\}^{-\alpha}\}^{|B(x,n)||B(x,n+1)|} \\ &\quad \geq \{1 - \beta(c_5 a^n)^{-\alpha}\}^{cb^{2n+1}} \to 1 \quad (n \to \infty). \end{aligned}$$

The parts of $k \geq 2$ are bounded from below by

$$\prod_{k=1}^{\infty} \left\{1 - \beta a^{-\alpha n}(a^{k-1} - 1)^{-\alpha}\right\}^{cb^{2n+k}} \equiv a_n,$$

and

$$\begin{aligned} \log a_n &\geq -c(b^2 a^{-\alpha})^n \sum_{k=2}^{\infty} (ba^{-\alpha})^k \\ &\to 0 \quad (n \to \infty) \end{aligned}$$

when $\alpha > 2D$, and the result follows. □

Remark 2.4. Theorems 1.3 and 1.4 only cover the cases such as the space like Cantor set. But we expect to hold the above under more general conditions (for example, finite ramified fractals like the Sierpinski gasket lattice). Also, whether percolation occurs or not in the region $D < \alpha \leq 2D$ is a remained problem.

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REFERENCES

1. M. Aizenman and C.M. Newman, Discontinuity of the percolation density in one dimensional $1/|x-y|^2$ Percolation Models. *Commun. Math. Phys.* **107**: 611–647 (1986).
2. P. Antal and A. Pisztora, On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* **24**: 1036–1048 (1996).
3. N. Berger, Transience, recurrence and critical behavior for long-range percolation. *Commun. Math. Phys.* **226**: 531–558 (2002).
4. N. Berger, A lower bound for the chemical distance in sparse long-range percolation models. Preprint (2004).
5. I. Benjamini and N. Berger, The diameter of long-range percolation clusters on finite cycles. *Random Struct. Algorithms* **19**(2):102–111 (2001).
6. I. Benjamini, H. Kesten, Y. Peres and O. Schramm, Geometry of the uniform spanning forest: Transitions in dimensions 4, 8, 12, ... *Ann. Math.* **160**: 465–491 (2004).
7. M. Biskup, Graph diameter in long-range percolation. Preprint (2004).
8. M. Biskup, On the scaling of the chemical distance in long-range percolation models. *Ann. Probab.* **32**: 2938–2977 (2004).
9. B. Bollobas, S. Janson and O. Riordan, Long-range percolation in $\mathbf{R}^d$. Preprint (2005).
10. D. Coppersmith, D. Gamarnik and M. Sviridenko, The diameter of a one-dimensional long-range percolation graph. *Random Struct. Algorithms* **21**: 1–13 (2002).
11. G. R. Grimmett, Percolation, 2nd edn. (Springer, 1999).
12. G. R. Grimmett, M. Keane and J. M. Marstrand. On the connectedness of a random graph. *Math. Proc. Camb. Philos. Soc.* **96**: 151–166 (1984).
13. S. Kalikow and B. Weiss, When are random graphs connected? *Israel J. Math.* **62**: 257–268 (1988).
14. H. Kesten, Correlation length and critical probabilities of slabs for percolation. Unpublished manuscript (1988).
15. T. Kumagai, Percolation on Pre-Sierpinski carpets. New Trend in Stochastic Analysis. *Proceedings of a Taniguchi International Workshop*, K. D. Elworthy et al., eds. (World Scientific, 1997), pp. 288–304.
16. C. M. Newman and L. S. Schulman, One dimensional $1/|j-i|^s$ percolation models: the existence of a transition for $s \leq 2$. *Commun. Math. Phys.* **104**: 547–571 (1986).

17. L. S. Schulman, Long range percolation in one dimension. *J. Phys. A. Lett.* **16**: L639–L641 (1983).
18. M. Shinoda, Existence of phase transition of percolation on Sierpinski carpet lattices. *J. Appl. Prob.* **39**: 1–10 (2002).
19. M. Shinoda, Non-existence of phase transition of oriented percolation on Sierpinski carpet lattices. *Prob. Theory Relat. Fields.* **125**: 447–456 (2003).